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Journal of Sound and Vibration 273 (2004) 1007–1029

JOURNAL OF SOUND AND VIBRATION

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Robust adaptive boundary control of an axially moving string under a spatiotemporally varying tension

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Received 3 December 2002; accepted 7 May 2003

Abstract

In this paper, a vibration suppression scheme for an axially moving string under a spatiotemporally varying tension and an unknown boundary disturbance is investigated. The lower bound of the tension variation is assumed to be sufficiently larger than the derivatives of the tension. The axially moving string system is divided into two spans, i.e., a controlled span and an uncontrolled span, by a hydraulic touch-roll actuator which is located in the middle section of the string. The transverse vibration of the controlled span part of the string is controlled by the hydraulic touch-roll actuator, and the position of the actuator is considered as the right boundary of the controlled span part. The mathematical model of the system, which consists of a hyperbolic partial differential equation describing the dynamics of the moving string and an ordinary differential equation describing the actuator dynamics, is derived by using the Hamilton's principle. The Lyapunov method is employed to design a robust boundary control law and adaptation laws for ensuring the vibration reduction of the controlled span part. The asymptotic stability of the closed loop system under the robust adaptive boundary control scheme is proved through the use of semigroup theory. Simulation results verify the effectiveness of the robust adaptive boundary controller proposed.

1. Introduction

Axially moving continuous materials can be found in various engineering areas: high-speed magnetic tapes, band saws, power transmission chains and belts, steel strips, and paper sheets

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0022-460X/03/$ - see front matter
doi:10.1016/S0022-460X(03)00519-4

under processing. Especially, the dynamics analysis and control for axially moving continuous materials have received a growing attention due to the entrance of new applications in flexible robotic manipulators and flexible space structures [1–4]. However, the utilization of the axially moving systems is limited because of the unwanted vibration of the systems in many applications, and in particular in high-speed precision systems. Support roller eccentricity, material non-uniformity, aero-dynamic disturbances, or manufacturing process (e.g., slitting, calendaring, printing, or galvanizing) can cause undesirable material vibration [5]. Active vibration control is then an important solution to reduce vibration and improve performance in many of the axial transport processes.

A boundary control to suppress the transverse vibration has several advantages over the control schemes acting within the spatial domain (e.g., distributed force control). A distributed controller is not practical to be implemented, and the distributed system becomes uncontrollable and unobservable when point actuators and sensors are located at nodal points [6]. A boundary control law, as opposed to a distributed control law, is not only easily implementable by active or semi-active means at the boundary, but also the dynamic model of the system equation is not altered by adding sensors and actuators. The boundary control law can also be derived from a Lyapunov function, which is related to the total mechanical energy based on the dynamics of the moving strip and sensing and actuation devices.

Vibration control schemes on axially moving strings include Refs. [7–13]. Those on axially moving beams include Refs. [14–16]. Particularly, in Ref. [12], a boundary controller based on adaptive computed-torque technique for an axially moving string system has been proposed. However, the implementation of the control scheme is not easy because the control law needs distributed data. In Ref. [13], a vibration isolation system together with a distributed axially moving string model has been presented, in which the controlled span has been shown to converge to zero asymptotically by an adaptive boundary controller. However, to achieve the asymptotic stability of the entire closed loop system, the slope on the uncontrolled side of the actuator should converge to zero as well. But, unfortunately the control law proposed does not satisfy this crucial condition due to the vibrations of the uncontrolled span part. Further, in all the papers mentioned above, the control laws have been designed under the assumption of a constant spatial tension. However, in practical situations, almost all axially moving systems have the varying tension that is a function of both time and space due to the eccentricity of a support roller, and/or external disturbances, and/or gravity, etc. [5,17]. Thus, to achieve a better control performance, a novel boundary controller incorporating the spatiotemporally varying tension has to be investigated.

In this paper, an axially moving string system which is divided into two spans, i.e., a controlled span and an uncontrolled span, by a transverse force actuator as shown in Fig. 1 is particularly focused. The main objective is to suppress the lateral vibrations in the controlled span with an implementable controller. The contributions of this paper are: A control-oriented string model for the axially moving continuous materials with a hydraulic touch-roll actuator is derived. The tension applied to the string is treated as a spatiotemporally varying function, which varies in unknown fashion but bounded. Considering practical situations, the lower bound of the tension variation is assumed to be sufficiently larger than the derivatives of the tension. Since the vibration of the uncontrolled span part acts as a disturbance to the actuator, a robust vibration suppression scheme is proposed. The mathematical models describing the dynamics of the moving string and the hydraulic touch-roll actuator for exerting boundary control force are represented as

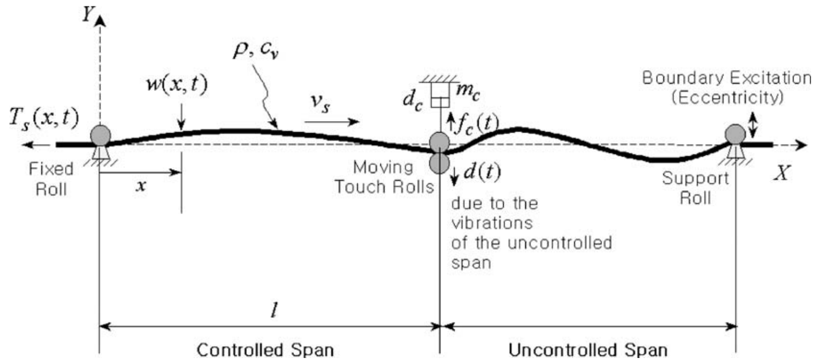

Fig. 1. A control-oriented schematic of the axially moving string with a hydraulic actuator.

a hyperbolic partial differential equation (PDE) and an ordinary differential equation (ODE), respectively. The coupled system of a PDE and an ODE is analyzed through the use of Lyapunov method and semigroup theory to derive a robust adaptive boundary controller and to ensure the stability of the closed loop system. The robust adaptive boundary controller proposed doesn't require distributed sensing and asymptotically stabilizes the controlled span part of the axially moving string system under the spatiotemporally varying tension while the vibrations of the uncontrolled span part remain.

The structure of this paper is organized as follows. In Section 2, the governing equation and boundary conditions of the axially moving string are derived by using Hamilton's principle. A boundary control problem is then formulated. In Section 3, a robust boundary force control with adaptation laws is proposed by using the Lyapunov method. In Section 4, the asymptotic stability of the closed loop system is investigated through the use of semigroup theory. In Section 5, computer simulations are provided. Conclusions are given in Section 6.

2. Problem formulation: equations of motion

Fig. 1 shows a schematic of an axially moving string for control system design purpose. The roll at the left boundary is assumed fixed, i.e., fixed in the sense that there is no vertical movement but it allows the string to move in the horizontal direction. The two touch rolls, where the control input (force) is exerted from the hydraulic actuator, are located in the middle section of the string. The touch rolls divide the string into two parts, i.e., a controlled span and an uncontrolled span as shown in Fig. 1. If only the controlled span part of the string is considered, the touch rolls play as the right boundary of the controlled span. Note that there appears a disturbance at the right boundary due to the vibration of the uncontrolled span of the string and the eccentricity of the support roll that causes a periodic excitation.

Let t be the time, x be the spatial co-ordinate along the longitude of motion, v_s be the axial speed of the string, $w(x,t)$ be the transversal displacement of the string at spatial co-ordinate x and time t, and l be the length of the controlled span from left to right boundaries. Also, let ρ be the mass per unit length, $T_s(x,t)$ be the tension applied to the string, and c_v be the viscous damping coefficient of the string. Let the mass and damping coefficients of the hydraulic actuator be m_c and d_c, respectively. The control force $f_c(t)$ is applied to the touch rolls to suppress the transverse vibrations of the controlled span part. $d(t)$ is the unknown but uniformly bounded external disturbance force at the right boundary due to the transverse vibration of the uncontrolled span.

Because the string travels with a constant speed v_s, the total derivative operator (material derivative) with respect to time should be defined as $\dot{(\cdot)} \triangleq \mathrm{d}(\cdot)/\mathrm{d}t = (\cdot)_t + v_s(\cdot)_x$ [9,12,16,18,19], where $(\cdot)_t = \partial(\cdot)/\partial t$ and $(\cdot)_x = \partial(\cdot)/\partial x$ denote the partial derivatives. Note that in this paper it is assumed that v_s belongs to a sub-critical speed since at a critical speed the fundamental natural frequency vanishes and divergence instability occurs [20].

The kinetic energy T, the potential energy V, and the virtual work δW by the external forces of the controlled span part between $x = 0$ and l including the hydraulic actuator are given as, respectively,

$$T = \frac{1}{2}\int_0^l \rho(v_s w_x + w_t)^2\,\mathrm{d}x + \frac{1}{2}m_c w_t^2(l,t), \tag{1}$$

$$V = \frac{1}{2}\int_0^l T_s(x,t) w_x^2\,\mathrm{d}x, \tag{2}$$

$$\delta W = f_c\delta w(l,t) - \int_0^l c_v(w_t + v_s w_x)\delta w\,\mathrm{d}x - d_c w_t(l,t)\delta w(l,t) - d(t)\delta w(l,t), \tag{3}$$

where $w_x = w_x(x,t)$ and $w_t = w_t(x,t)$ have been used, and similar abbreviations will be used in the sequel.

Now, by using the Hamilton's principle, i.e., $\int_{t_0}^{t_1}(\delta T - \delta V + \delta W)\,\mathrm{d}t = 0$ [19, p. 256], the governing equation and boundary conditions of the controlled span part of the axially moving string are derived as follows:

$$\begin{aligned}&\rho w_{tt}(x,t) + 2\rho v_s w_{xt}(x,t) + \rho v_s^2 w_{xx}(x,t) - (T_s(x,t)w_x(x,t))_x \\ &\quad + c_v(w_t(x,t) + v_s w_x(x,t)) = 0, \quad 0 < x < l,\end{aligned} \tag{4}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_{t0}(x), \tag{5}$$

$$w(0,t) = 0, \tag{6}$$

and

$$f_c(t) = m_c w_{tt}(l,t) + (d_c - \rho v_s)w_t(l,t) + (T_s(l,t) - \rho v_s^2)w_x(l,t) + d(t). \tag{7}$$

Eq. (4) is a linear PDE governing the transverse motion $w(x,t)$ of the axially moving string. Eqs. (5) and (6)–(7) denote the initial conditions and the boundary conditions of the controlled span part of the string, respectively. Note that the right boundary condition (7) is an ODE that describes the motion of the hydraulic actuator in compliance with the transversal force at $x = l$. For notational brevity, all variables at boundaries will be written without explicit time in the sequel, i.e., $w(0) = w(0,t)$, $w(l) = w(l,t)$, $T_s(l) = T_s(l,t)$, etc.

As shown in the above equations, the tension $T_s(x,t)$ is a spatiotemporally varying function, which varies in unknown fashion but bounded. If the string is moving vertically, the gravitational force $\rho g x$, which acts as an additional tension to the string, may not be neglected [5]. Also the tension itself may be time-varying due to the eccentricity of a support roller, which causes a periodic excitation. Thus, the tension variation $T_s(x,t)$ in the string has to be incorporated in control law design. But, it is assumed that $T_s(x,t)$ is sufficiently smooth and uniformly bounded as follows:

$$0 < T_{s,min} \leqslant T_s(x,t) \leqslant T_{s,max}, \tag{8}$$

$$|(T_s(x,t))_t| \leqslant (T_s)_{t,max}, \tag{9}$$

$$|(T_s(x,t))_x| \leqslant (T_s)_{x,max}, \tag{10}$$

for all $x \in [0,l]$, $t \geqslant 0$, and some a priori known constants $T_{s,min}$, $T_{s,max}$, $(T_s)_{t,max}$, and $(T_s)_{x,max}$. Considering practical situations, it is also assumed that the lower bound $T_{s,min}$ is sufficiently larger than both $(T_s)_{t,max}$ and $(T_s)_{x,max}$.

The control objective is now to stabilize asymptotically the vibration energy of the controlled span part with a time-varying condition at $x = l$. From Eqs. (1)–(2), the string energy $E_m(t)$ of the controlled span part is given by

$$E_m(t) = \frac{1}{2}\int_0^l \rho(v_s w_x + w_t)^2\,\mathrm{d}x + \frac{1}{2}\int_0^l T_s w_x^2\,\mathrm{d}x, \tag{11}$$

where the traveling speed of the string, v_s, is a constant and $T_s = T_s(x,t)$.

In the remainder of this section, in order to provide a specific idea regarding how a boundary control works, the dynamics of the moving string with fixed boundaries but still allowing an axial movement is first analyzed. The time derivative of $E_m(t)$ in Eq. (11) is now evaluated by applying the one-dimensional transport theorem of moving material (see, Refs. [9,11,21, p. 214]) as follows:

$$\begin{aligned}\dot{E}_m(t) = &\int_0^l \rho(w_t + v_s w_x)(w_{tt} + 2v_s w_{xt} + v_s^2 w_{xx})\,\mathrm{d}x \\ &+ \int_0^l T_s w_x(w_{xt} + v_s w_{xx})\,\mathrm{d}x + \frac{1}{2}\int_0^l \{(T_s)_t + v_s(T_s)_x\} w_x^2\,\mathrm{d}x \\ = &\int_0^l (w_t + v_s w_x)((T_s w_x)_x - c_v(w_t + v_s w_x))\,\mathrm{d}x \\ &+ \int_0^l T_s w_x(w_{xt} + v_s w_{xx})\,\mathrm{d}x + \frac{1}{2}\int_0^l \{(T_s)_t + v_s(T_s)_x\} w_x^2\,\mathrm{d}x,\end{aligned} \tag{12}$$

where Eq. (4) has been used in deriving the second equality. Integrating by parts, the terms in Eq. (12) are further simplified as follows:

$$\int_0^l \{w_t(T_s w_x)_x + T_s w_x w_{xt}\}\,\mathrm{d}x = [w_t(T_s w_x)]_0^l, \tag{13}$$

$$\int_0^l w_x(T_s w_x)_x\,\mathrm{d}x = [T_s w_x^2]_0^l - \int_0^l T_s w_x w_{xx}\,\mathrm{d}x. \tag{14}$$

The substitution of Eqs. (13)–(14), together with the fixed boundary conditions, into Eq. (12) yields:

$$\dot{E}_m(t) = -c_v \int_0^l (w_t + v_s w_x)^2 \,\mathrm{d}x - v_s T_s(0) w_x^2(0) + v_s T_s(l) w_x^2(l) + \frac{1}{2}\int_0^l \{(T_s)_t + v_s (T_s)_x\} w_x^2 \,\mathrm{d}x. \tag{15}$$

Now, we make the following observations from Eq. (15). (i) The viscous damping reduces the mechanical energy of the axially moving string, which can be seen from the existence of $-c_v \int_0^l (w_t + v_s w_x)^2 \,\mathrm{d}x$. (ii) Even though the transverse displacements at both boundaries are zero, the transverse force due to instantaneous string tension, $v_s T_s w_x$, at each boundary can still deform the string [9]. (iii) The time derivative of $T_s(x,t)$, i.e., $(T_s)_t + v_s(T_s)_x$, increases the energy flux by the factor of w_x^2. The last observation fully justifies that the time rate of the change of $T_s(x,t)$ cannot be simply neglected.

Thus, it is concluded that for the traveling string with even fixed boundary conditions, the string tension at the right boundary $x = l$ and the time rate of the change of tension $T_s(x,t)$ should be properly included to decrease the mechanical energy $E_m(t)$. Note that if $(T_s)_t + v_s(T_s)_x$ and time-varying boundary conditions are uniformly bounded, then it can be concluded that $w(x,t)$ and $w_t(x,t)$ for $0 \leqslant x \leqslant l$ are uniformly ultimately bounded, which will be proved via Section 3 and Section 4 next. Therefore, it can be easily concluded that the strip displacement and velocity of the uncontrolled span part are also uniformly ultimately bounded if time-varying conditions at the left and right boundaries of the uncontrolled span part, i.e., the hydraulic touch-roll actuator and the support roll, are bounded. Since w_x and w_t at the support roll with a periodic excitation are really bounded, only the boundedness of those at the actuator is then needed and that will be proved through the stability analysis in Section 4. However, the displacement of the uncontrolled span part may not converge to zero due to the periodic excitation at the support roller. Thus, such undesired vibrations of the uncontrolled span part give an effect to the hydraulic actuator like an external force. In this paper, the unknown but bounded external force applied to the actuator is defined by $d(t)$ and is treated as a right boundary disturbance on the controlled span part, and then the stability analysis of the axially moving string system is done by considering only the controlled span part of the string.

3. Design of robust adaptive boundary control laws

In this section, a robust boundary control law and adaptation laws to suppress the vibration energy of the controlled span part are derived using the Lyapunov method. The selection of a suitable Lyapunov function candidate and the construction of an effective control law are the most important issues in the Lyapunov method. As shown in Eqs. (4)–(7), the control mechanism is coupled to the string system since the control system is attached to the boundary of the string from which the control force f_c is applied. Hence, to obtain the asymptotic stability of coupled system (4)–(7), the convergence of both the hydraulic actuator displacement $w(l)$ and the velocity $w_t(l)$ to zero should also be satisfied. But, the total mechanical energy $E_m(t) + m_c w_t^2(l)/2$ from

Eqs. (1)–(2) does not involve $w(l)$ of the hydraulic actuator. Thus, a modification (11) is pursued to get an appropriate Lyapunov function candidate of coupled system (4)–(7).

Note that by boundary condition (6) and Poincare's inequality [22, p. 67], there exists a positive constant C_1 such that

$$\int_0^l w^2\,\mathrm{d}x \leqslant C_1 \int_0^l w_x^2\,\mathrm{d}x \quad \text{for } x\in[0,l]. \tag{16}$$

From Eq. (16), it is seen that the convergence of the transverse displacement $w(x,t)$ for $x\in[0,l]$ to zero is assured by considering the convergence of the slope $w_x(x,t)$ for $x\in[0,l]$ to zero. That is, if $\lim_{t\to\infty}\int_0^l w_x^2\,\mathrm{d}x = 0$ is satisfied, then $\lim_{t\to\infty}\int_0^l w^2\,\mathrm{d}x = 0$ is also satisfied. Thus, the asymptotic stability of the hydraulic actuator can be analyzed by adding the slope term at $x = l$ in the mechanical energy, i.e.,

$$V_m(t) = E_m(t) + \frac{1}{2}m_c\{w_t^2(l) + w_x^2(l)\}. \tag{17}$$

Assuming that the disturbance $|d(t)|$ is uniformly bounded by μ_d, i.e., $\mu_d \geqslant |d(t)|$, where μ_d is an unknown positive constant, the following positive definite functional $V(t)$, replacing $V_m(t)$ in Eq. (17), is now considered:

$$\begin{aligned} V(t) &= \alpha V_0(t) + 2\beta \int_0^l \rho x w_x(w_t + v_s w_x)\,\mathrm{d}x + \frac{1}{2\gamma_d}\tilde{\mu}_d^2 \\ &= \alpha V_0(t) + \beta \int_0^l \rho x\{w_x + (w_t + v_s w_x)\}^2\,\mathrm{d}x - \beta \int_0^l \rho x w_x^2\,\mathrm{d}x \\ &\quad - \beta \int_0^l \rho x(w_t + v_s w_x)^2\,\mathrm{d}x + \frac{1}{2\gamma_d}\tilde{\mu}_d^2, \end{aligned} \tag{18}$$

where α and β are positive constants,

$$V_0(t) = \frac{1}{2}\int_0^l \rho(v_s w_x + w_t)^2\,\mathrm{d}x + \frac{1}{2}\int_0^l T_s w_x^2\,\mathrm{d}x + \frac{1}{2\alpha}m_c\,\{\alpha w_t(l) + (\alpha v_s + 2\beta l)w_x(l)\}^2,$$

$$\frac{\alpha}{2}\int_0^l \rho(w_t + v_s w_x)^2\,\mathrm{d}x \geqslant \beta \int_0^l \rho l(w_t + v_s w_x)^2\,\mathrm{d}x \geqslant \beta \int_0^l \rho x(w_t + v_s w_x)^2\,\mathrm{d}x,$$

$$\frac{\alpha}{2}\int_0^l T_s(x,t)w_x^2\,\mathrm{d}x \geqslant \frac{\alpha}{2}T_{s,min}\int_0^l w_x^2\,\mathrm{d}x \geqslant \beta \int_0^l \rho l w_x^2\,\mathrm{d}x \geqslant \beta \int_0^l \rho x w_x^2\,\mathrm{d}x,$$

$$\tilde{\mu}_d = \hat{\mu}_d - \mu_d,$$

and $\hat{\mu}_d$ is the adaptive estimate of μ_d, which will be specified in the sequel. From Eq. (18), it is seen that $V(t)$ is a positive definite functional if $\alpha > 2\beta l$ and $T_{s,min} > \rho$.

By using Cauchy–Schwarz inequality, the following holds:

$$(\alpha - C_2)V_0(t) \leqslant V^*(t) \leqslant (\alpha + C_2)V_0(t), \tag{19}$$

where $V^*(t) \triangleq V(t) - (1/(2\gamma_d))\tilde{\mu}_d^2$ and $C_2 = 2\beta l$. From Eq. (19), it can be concluded that $V^*(t)$ is equivalent to the positive definite functional $V_0(t)$ which is also equivalent to $V_m(t)$ in Eq. (17).

As a result, the positive definite functional $V(t)$ given by Eq. (18) is considered as a Lyapunov function candidate of coupled system (4)–(7).

3.1. Control laws under unknown disturbance

A robust adaptive boundary force controller is designed to asymptotically stabilize the controlled span part of the axially moving string in the presence of the spatiotemporally varying tension and the unknown boundary disturbance while all signals in the closed loop remain bounded.

The time derivative of $V(t)$ along Eq. (4) yields

$$\begin{aligned}\dot{V}(t) &= \alpha\dot{V}_0 + \frac{\mathrm{d}}{\mathrm{d}t}\left[2\beta\int_0^l \rho x w_x(w_t + v_s w_x)\,\mathrm{d}x\right] + \frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d \\ &= \alpha\dot{V}_0(t) + 2\beta\rho\int_0^l x(w_{xt} + v_s w_{xx})(w_t + v_s w_x)\,\mathrm{d}x \\ &\quad + 2\beta\rho\int_0^l x w_x(w_{tt} + 2v_s w_{xt} + v_s^2 w_{xx})\,\mathrm{d}x + \frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d \\ &= \alpha\dot{V}_0(t) + \beta\rho[x(w_t + v_s w_x)^2]_0^l - \beta\rho\int_0^l (w_t + v_s w_x)^2\,\mathrm{d}x \\ &\quad + 2\beta\int_0^l x w_x((T_s w_x)_x - c_v(w_t + v_s w_x))\,\mathrm{d}x + \frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d.\end{aligned} \tag{20}$$

The integration by parts yields:

$$2\int_0^l x w_x(T_s w_x)_x\,\mathrm{d}x = [x(T_s w_x^2)]_0^l - \int_0^l T_s w_x^2\,\mathrm{d}x + \int_0^l x(T_s)_x w_x^2\,\mathrm{d}x. \tag{21}$$

The following inequality is also utilized:

$$uv \leqslant \gamma u^2 + \frac{1}{\gamma}v^2 \quad \text{for any } \gamma > 0. \tag{22}$$

Using Eq. (22), the following are derived:

$$\int_0^l x w_x(w_t + v_s w_x)\,\mathrm{d}x \leqslant l\gamma_1\int_0^l w_x^2\,\mathrm{d}x + \frac{l}{\gamma_1}\int_0^l (w_t + v_s w_x)^2\,\mathrm{d}x, \quad \gamma_1 > 0. \tag{23}$$

Thus, by substituting Eqs. (4), (6)–(7), (8)–(10), (13)–(14), and (21)–(23) into Eq. (20), the time derivative of the Lyapunov function candidate $V(t)$, with a right boundary actuator at $x = l$ and the fixed condition at left boundary, becomes

$$
\begin{aligned}
\dot{V}(t) \leqslant & -\left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right) \int_0^l (w_t + v_s w_x)^2 \, \mathrm{d}x \\
& - \left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\} \int_0^l w_x^2 \, \mathrm{d}x \\
& - \alpha v_s T_s(0) w_x^2(0) + \alpha T_s(l) w_t(l) w_x(l) + (\alpha v_s + \beta l) T_s(l) w_x^2(l) \\
& + \beta\rho l (w_t(l) + v_s w_x(l))^2 + \frac{1}{\gamma_d} \tilde{\mu}_d \dot{\hat{\mu}}_d + \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\} \\
& \times \left\{f_c + d - (d_c - \rho v_s) w_t(l) + \rho v_s^2 w_x(l) + \frac{m_c}{\alpha}(\alpha v_s + 2\beta l) w_{xt}(l)\right\} \\
& - \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\} T_s(l) w_x(l).
\end{aligned} \tag{24}
$$

A robust boundary force control law is then proposed as follows:

$$
f_c = (d_c - \rho v_s) w_t(l) - \rho v_s^2 w_x(l) - \frac{m_c}{\alpha}(\alpha v_s + 2\beta l) w_{xt}(l) + f_d + k_1 w_x(l) - k_2 w_t(l), \tag{25}
$$

where

$$
0 < k_1 < \frac{T_s(l)\beta l - \beta\rho l v_s^2}{\alpha v_s + 2\beta l}, \quad k_2 > \frac{\beta\rho l}{2\alpha}, \quad \text{and} \quad k_2 = \frac{k_1\alpha + 2\beta\rho l v_s}{\alpha v_s + 2\beta l}, \tag{26}
$$

and the additional term $f_d(t)$ is regarded as a new input signal to be determined based on robust control strategy and is given by

$$
f_d(t) = -\frac{\hat{\mu}_d^2(t)}{\hat{\mu}_d(t)|\bar{w}(l)| + \varepsilon_d} \bar{w}(l), \tag{27}
$$

where $\bar{w}(l) = \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\}$ and $\varepsilon_d > 0$. The adaptation law $\hat{\mu}_d$ in Eq. (27) is proposed as follows:

$$
\dot{\hat{\mu}}_d(t) = -\delta_d \hat{\mu}_d(t) + \gamma_d |\bar{w}(l)|, \tag{28}
$$

where $\delta_d > 0$ and $\gamma_d > 0$. The term $-\delta_d \hat{\mu}_d$ in Eq. (28) is purposely inserted to ensure that $\hat{\mu}_d$ does not become unbounded.

The substitution of Eqs. (25), (27), and adaptation law (28) into Eq. (24) yields

$$
\begin{aligned}
\dot{V}(t) \leqslant & -\left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right) \int_0^l (w_t + v_s w_x)^2 \mathrm{d}x \\
& - \left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\} \int_0^l w_x^2 \, \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha v_s T_s(0) w_x^2(0) + \beta\rho l (w_t(l) + v_s w_x(l))^2 - T_s(l)\beta l w_x^2(l) + \{k_1 w_x(l) - k_2 w_t(l)\} \\
&\times \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\} + \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\}(f_d + d) + \frac{1}{\gamma_d}\tilde{\mu}_d \dot{\hat{\mu}}_d \\
\leqslant & - \left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right) \int_0^l (w_t + v_s w_x)^2 \mathrm{d}x \\
&- \left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\} \int_0^l w_x^2 \,\mathrm{d}x \\
&- \alpha v_s T_s(0) w_x^2(0) - \{T_s(l)\beta l - k_1(\alpha v_s + 2\beta l) - \beta\rho l v_s^2\} w_x^2(l) \\
&- (2k_2\alpha - \beta\rho l) w_t^2(l) + \{2\beta\rho l v_s + k_1\alpha - k_2(\alpha v_s + 2\beta l)\} w_x(l) w_t(l) \\
&- \frac{\delta_d}{\gamma_d}\tilde{\mu}_d^2 + \varepsilon_d + \frac{\delta_d}{2\gamma_d}\mu_d,
\end{aligned} \tag{29}
$$

where $\{T_s(l)\beta l - k_1(\alpha v_s + 2\beta l) - \beta\rho l v_s^2\} > 0$, $\{2\beta\rho l v_s + k_1\alpha - k_2(\alpha v_s + 2\beta l)\} > 0$ and $(2k_2\alpha - \beta\rho l) > 0$ by Eq. (26). Note that in deriving the second inequality in Eq. (29), the following relationship with detailed derivation in Appendix B has been utilized:

$$
\{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\}(f_d + d) + \frac{1}{\gamma_d}\tilde{\mu}_d \dot{\hat{\mu}}_d \leqslant -\frac{\delta_d}{\gamma_d}\tilde{\mu}_d^2 + \varepsilon_d + \frac{\delta_d}{2\gamma_d}\mu_d. \tag{30}
$$

Since $T_{s,min}$ is sufficiently large, the positive values α, β, and γ_i, $i = 1, 2$, can be chosen to satisfy

$$
\left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right) > 0, \tag{31}
$$

$$
\left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\} > 0. \tag{32}
$$

Thus, the following is obtained:

$$
\dot{V}(t) \leqslant -C_3\left(\int_0^l (w_t + v_s w_x)^2 \,\mathrm{d}x + \int_0^l w_x^2 \,\mathrm{d}x + w_t^2(l) + w_x^2(l)\right) + \upsilon(t), \tag{33}
$$

where

$$
\begin{aligned}
C_3 = \min\Big\{ &\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1},\ \beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l, \\
&T_s(l)\beta l - k_1(\alpha v_s + 2\beta l) - \beta\rho l v_s^2,\ 2k_2\alpha - \beta\rho l \Big\},
\end{aligned}
$$

$\upsilon(t) \triangleq -(\delta_d/\gamma_d)\tilde{\mu}_d^2 + \varepsilon_d + (\delta_d/2\gamma_d)\mu_d$, and $\upsilon(t)$ is bounded because of the assumption that μ_d is bounded.

Remark 1. $\upsilon(t)$ in Eq. (33) can be pushed in an arbitrarily small boundedness region by making sufficiently small ε_d, δ_d and sufficiently large γ_d.

3.2. Control laws under unknown parameter and disturbance

An adaptive control law for the case of unknown material density is now considered. The robust boundary force control law (25) depends on exact knowledge of the system and actuator parameters, i.e., ρ, m_c, and d_c. The values of actuator parameters, m_c and d_c, can be determined with certainty by the designer of the actuator, but not that of system parameter ρ. Thus, to compensate for the unknown constant or slowly time-varying parameter ρ, an adaptive control law is needed and from which the unknown parameter is estimated and used to update the robust boundary control law.

From Eq. (25), the robust boundary control law is rewritten using an estimated parameter $\hat{\rho}$ as follows:

$$f_c = (d_c - \hat{\rho}v_s)w_t(l) - \hat{\rho}v_s^2 w_x(l) - \frac{m_c}{\alpha}(\alpha v_s + 2\beta l)w_{xt}(l) + f_d \\ + k_1 w_x(l) - k_2 w_t(l), \tag{34}$$

where $\hat{\rho}$ is the adaptive estimate of ρ and the adaptation law is given by

$$\dot{\hat{\rho}} = \gamma_0 v_s\{w_t(l) + v_s w_x(l)\}\{\alpha w_t(l) + (\alpha v_s + 2\beta l)w_x(l)\}, \tag{35}$$

where $\gamma_0 > 0$ is the adaptation gain.

The substitution of Eq. (34) into Eq. (7) yields

$$m_c w_{tt}(l) = -\frac{m_c}{\alpha}(\alpha v_s + 2\beta l)w_{xt}(l) - \tilde{\rho}v_s w_t(l) - (T_s(l) + \tilde{\rho}v_s^2)w_x(l) + f_d - d \\ + k_1 w_x(l) - k_2 w_t(l), \tag{36}$$

where $\tilde{\rho} = \hat{\rho} - \rho$ is the parameter estimation error. By applying Eq. (18), define a positive definite functional $\hat{V}(t)$ which is another Lyapunov function candidate such that

$$\hat{V}(t) = \alpha V_0(t) + 2\beta \int_0^l \rho x w_x(w_t + v_s w_x)\,\mathrm{d}x + \frac{1}{2\gamma_d}\tilde{\mu}_d^2 + \frac{1}{2\gamma_0}\tilde{\rho}^2. \tag{37}$$

Differentiating Eq. (37) with respect to t along Eqs. (20)–(29) yields

$$\dot{\hat{V}}(t) \leqslant -\left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right)\int_0^l (w_t + v_s w_x)^2 \mathrm{d}x \\ - \left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\}\int_0^l w_x^2\,\mathrm{d}x$$

$$
\begin{aligned}
&- \alpha v_s T_s(0) w_x^2(0) - \left\{ T_s(l)\beta l - k_1(\alpha v_s + 2\beta l) - \beta\rho l v_s^2 \right\} w_s^2(l) - (2k_2\alpha - \beta\rho l) w_t^2(l) \\
&- \frac{\delta_d}{\gamma_d}\tilde{\mu}_d^2 + \varepsilon_d + \frac{\delta_d}{2\gamma_d}\mu_d \\
&+ \frac{1}{\gamma_0}\tilde{\rho}[\dot{\tilde{\rho}} - \gamma_0 v_s\{w_t(l) + v_s w_x(l)\}\{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\}]. \qquad (38)
\end{aligned}
$$

The substitution of adaptation law (35) into Eq. (38) yields the same result as Eq. (33).

Remark 2. In implementing robust boundary force control law (34) and adaptation laws (28) and (35), measurement of the velocity $w_t(l)$, slope $w_x(l)$, and slope rate $w_{xt}(l)$ at $x = l$ on the controlled side of the actuator are required. By using an encoder (or photodiode) on the actuator and two laser sensors, the actuator displacement $w(l)$ and the slope $w_x(l)$ on the controlled side of the actuator can be measured, respectively (see Ref. [13]). Filtered backwards differencing of the signals provides the actuator velocity $w_t(l)$ and the slope rate $w_{xt}(l)$, respectively.

Remark 3. $\dot{V}(t)$ in Eq. (33) and $\dot{\hat{V}}(t)$ in Eq. (38) may take positive values because of the last term $\upsilon(t)$. It implies that the right boundary disturbance on the controlled span part due to the transverse vibration of the uncontrolled part causes an increase in the mechanical energy of the controlled span part of the string. Thus, no stability conclusion can be drawn from the Lyapunov function candidates $V(t)$ and $\hat{V}(t)$. But in next section it is shown that the robust adaptive boundary controller, (34), (35), and (28), assures the boundedness of all signals in the closed loop system and the convergence near to zero.

4. Stability analysis

In this section, the asymptotic stability of the controlled span part of the axially moving system under robust boundary control law (34) and adaptation laws (28) and (35) is proven. Eq. (36) is rewritten as

$$
\begin{aligned}
w_{tt}(l) = &- m_c^{-1}(T_s(l) - k_1)w_x(l) + v_s\alpha^{-1}(\alpha v_s + 2\beta l)w_{xx}(l) \\
&- \alpha^{-1}(\alpha v_s + 2\beta l)(w_{xt}(l) + v_s w_{xx}(l)) \\
&- m_c^{-1}k_2 w_t(l) - m_c^{-1}\,\tilde{\rho} v_s(w_t(l) + v_s w_x(l)) + m_c^{-1}(f_d - d). \qquad (39)
\end{aligned}
$$

In other to analyze the asymptotic stability of the closed loop system (4)–(6) and (39), the state space $\Im$ is defined as follows:

$$\Im \triangleq \{(w, \dot{w}, w(l), w_t(l))^{\mathrm{T}} | w \in H_L^2, w_t \in L^2, \text{ and } w(l), w_t(l) \in R\}, \qquad (40)$$

where the superscript T stands for transpose. The spaces L_2 and H_L^k are defined as follows:

$$L^2 \triangleq \left\{ f : [0, l] \to R \,\middle|\, \int_0^l f^2\,\mathrm{d}x < \infty \right\}, \qquad (41)$$

$$H_L^k \triangleq \left\{ f \in L^2 \left| \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots, \frac{\partial^k f}{\partial x^k} \in L^2, \text{ and } f(0) = 0 \right. \right\}. \tag{42}$$

In the space $\Im$, the inner-product is defined as follows:

$$\begin{aligned}\langle z, z' \rangle_{\Im} \triangleq & \frac{\alpha}{2} \int_0^l T_s w_x w'_x \, \mathrm{d}x + \frac{\alpha}{2} \int_0^l \rho \dot{w} \dot{w}' \mathrm{d}x \\ & + \beta \int_0^l \rho x (w_x + \dot{w})(w'_x + \dot{w}') \mathrm{d}x - \beta \int_0^l \rho x w_x w'_x \, \mathrm{d}x - \beta \int_0^l \rho x \dot{w} \dot{w}' \mathrm{d}x \\ & + \frac{1}{2} m_c \{(\alpha v_s + 2\beta l) w_x(l) + \alpha w_t(l)\} \{(\alpha v_s + 2\beta l) w'_x(l) + \alpha w'_t(l)\} \\ = & \frac{\alpha}{2} \int_0^l T_s w_x w'_x \, \mathrm{d}x + \frac{\alpha}{2} \int_0^l \rho \dot{w} \dot{w}' \mathrm{d}x + \beta \int_0^l \rho x w_x \dot{w}' \mathrm{d}x + \beta \int_0^l \rho x \dot{w} w'_x \, \mathrm{d}x \\ & + \frac{1}{2} m_c \{(\alpha v_s + 2\beta l) w_x(l) + \alpha w_t(l)\} \{(\alpha v_s + 2\beta l) w'_x(l) + \alpha w'_t(l)\},\end{aligned} \tag{43}$$

where $z = (w, \dot{w}, w(l), w_t(l))^{\mathrm{T}}$, $z' = (w', \dot{w}', w'(l), w'_t(l))^{\mathrm{T}} \in \Im$, and $\alpha, \beta > 0$.

Define a state space $W \triangleq \Im \times R^2$ for the closed loop system including the two adaptation laws. By using

$$\ddot{w} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} w = \frac{\mathrm{d}}{\mathrm{d}t}(w_t + v_s w_x) = w_{tt} + 2 v_s w_{xt} + v_s^2 w_{xx},$$

the coupled dynamics (4)–(6) and (39), (35), and (28) can then be rewritten as

$$\dot{y} = A_0(t) y + F_0(t, y), \quad y(0) \in W, \tag{44}$$

where $y = (z, \hat{\rho}, \hat{\mu}_d)^{\mathrm{T}} = (w, \dot{w}, w(l), w_t(l), \hat{\rho}, \hat{\mu}_d)^{\mathrm{T}} \in W$. Let

$$\begin{aligned} D_0 \triangleq & \{(w, \dot{w}, w(l), w_t(l), \hat{\rho}, \hat{\mu}_d)^{\mathrm{T}} \mid w \in H_L^2, w_t \in H_L^1, w(l), w_t(l), \hat{\rho}, \hat{\mu}_d \in R, \ w(0) = 0, \text{ and} \\ & m_c w_{tt}(l) + \frac{m_c}{\alpha}(\alpha v_s + 2\beta l) w_{xt}(l) + \tilde{\rho} v_s w_t(l) + \left(T_s(l) - \tilde{\rho} v_s^2\right) w_x(l) - f_d + d - k_1 w_x(l) + k_2 w_t(l) = 0\} \end{aligned} \tag{45}$$

and define a family of operators $\{A_0(t)\}_{t \geqslant 0}$, $A_0 : D_0 \subset W \to W$ by

$$A_0(t) \triangleq \begin{bmatrix} A(t) & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & -\delta_d \end{bmatrix} \tag{46}$$

and $F_0(t, y)$ by

$$F_0(t, y) \triangleq \begin{bmatrix} F(t, z, \hat{\rho}, \hat{\mu}_d) \\ \varepsilon \hat{\rho} + \gamma_0 v_s \{w_t(l) + v_s w_x(l)\} \{\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)\} \\ \gamma_d |\alpha w_t(l) + (\alpha v_s + 2\beta l) w_x(l)| \end{bmatrix}, \tag{47}$$

where $\varepsilon > 0$ and

$$A(t) \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ \rho^{-1}\dfrac{\partial}{\partial x}\left(T_s\dfrac{\partial}{\partial x}\right) & -\rho^{-1}c_v & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \left\{\begin{aligned} & -m_c^{-1}(T_s - k_1)\dfrac{\partial}{\partial x}\Big|_{x=l} \\ & +\dfrac{v_s}{\alpha}(\alpha v_s + 2\beta l)\dfrac{\partial^2}{\partial x^2}\Big|_{x=l} \end{aligned}\right\} & -\dfrac{1}{\alpha}(\alpha v_s + 2\beta l)\dfrac{\partial}{\partial x}\Big|_{x=l} & 0 & -m_c^{-1}k_2 \end{bmatrix}$$

and

$$F(t, z, \hat{\rho}, \hat{\mu}_d) \triangleq \begin{bmatrix} 0 \\ 0 \\ 0 \\ -m_c^{-1}\tilde{\rho}v_s(w_t(l) + v_s w_x(l)) + m_c^{-1}(f_d - d) \end{bmatrix}.$$

The operator $A_0(t)$ given by Eq. (46) is dissipative in the space W because

$$\langle A_0(t)y, y\rangle_W \leqslant -C_3\left(\int_0^l (w_t + v_s w_x)^2\,\mathrm{d}x + \int_0^l w_x^2\,\mathrm{d}x + w_t^2(l) + w_x^2(l)\right) - \varepsilon\hat{\rho}^2 - \delta_d\hat{\mu}_d^2 \leqslant 0,$$

where C_3 is the positive constant defined in Eq. (33). The operator $A_0(t)$ then satisfies the following conditions: (i) There exists $\rho_1 > 0$ such that $|\langle A_0(t)y, y'\rangle_W| \leqslant \rho_1\|y\|_W\|y'\|_W$, $y, y' \in W$. (ii) There exists $\rho_2 > 0$ for which $\langle A_0(t)y, y\rangle_W \leqslant -\rho_2\|y\|_W^2$, $y \in W$. The above conditions (i) and (ii) imply that the operator $A_0(t)$ restricted to the subspace D_0 is the infinitesimal generator of an analytic semigroup of bounded linear operators on W (see Ref. [23]). Therefore, $\{A_0(t)\}_{t\geqslant 0}$ is a stable family of infinitesimal generators of C_0 semigroups on W (see Ref. [24, p. 130]). Furthermore, the map $t \to A_0(t)y$ for $y \in D_0$ is strongly continuously differentiable in W. Note that $F_0 : W \to W$ is locally Lipschitz continuous in W, i.e., $\|F_0(t, y) - F_0(t, y')\|_W \leqslant C_4\|y - y'\|_W$, where $y, y' \in W$, and C_4 is a positive constant (see Appendix C). Thus, it follows from Theorem 5.3 in Ref. [24, p. 147] that there exists a unique evolution system, $\{S(t, \tau) : 0 \leqslant \tau \leqslant t\}$, on W associated with the homogeneous system corresponding to Eq. (44). Finally, the solution $z(t)$ of system (44) can be written as

$$z(t) = \Phi(t, 0)z_0 + \int_0^t \Phi(t, \tau)F(\tau, z(\tau), \hat{\rho}(\tau), \hat{\mu}_d(\tau))\,\mathrm{d}\tau, \quad t \geqslant 0, \tag{48}$$

where $\Phi(t, \tau)$ is the evolution operator associated with $A(t)$ in the space $\mathfrak{I}$.

The behavior of the solution $z(t)$ above is summarized as follows:

Theorem 1. *Consider the closed loop system consisting of plant and actuator dynamics* (4)–(6), (39) *and adaptive estimators* (28) *and* (35). *Suppose that the following hold*:

$$\alpha > 2\beta l, \quad T_{s,min} > \rho, \quad \left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right) > 0,$$

$$\left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\} > 0,$$

$$0 < k_1 < \frac{T_s(l)\beta l - \beta\rho l v_s^2}{\alpha v_s + 2\beta l}, \quad k_2 > \frac{\beta\rho l}{2\alpha}, \quad \text{and} \quad k_2 = \frac{k_1\alpha + 2\beta\rho l v_s}{\alpha v_s + \beta l},$$

and $\upsilon(t)$ *in Eq.* (33) *is uniformly bounded by a positive constant* σ. *Then,*

(i) *The solutions of closed loop system and adaptation laws, i.e.,* $z(t)$, $\hat{\rho}(t)$, *and* $\hat{\mu}_d(t)$, *are uniformly ultimately bounded.*
(ii) *The uniform ultimate boundedness region of* $z(t)$ *can be made arbitrarily small near to zero by a suitable choice of* ε_d, δ_d, *and* γ_d.
(iii) *Furthermore, if* $\int_0^\infty \upsilon(t)\,dt < \infty$, $z(t) \to 0$ *as* $t \to \infty$.

Proof. To prove assertions (i)–(iii), the following are made:

(A1) For all $t \geqslant 0$

$$\|F(t, z, \hat{\rho}, \hat{\mu}_d)\|_{\Im} \leqslant \alpha_0(\theta(t))\|z(t)\|_{\Im} + c_0, \tag{49}$$

where c_0 is a positive constant and $\alpha_0 : R^2 \to R^+$ is bounded for finite value of $\theta(t) \triangleq (\hat{\rho}, \hat{\mu}_d)^{\mathrm{T}}$.

(A2) The functional $\hat{V}(t) : R^+ \to R^+$ given by Eq. (37) satisfies

$$\alpha_1(\|z(t)\|_{\Im}) + \beta_1(|\theta(t)|) \leqslant \hat{V}(t) \leqslant \alpha_2(\|z(t)\|_{\Im}) + \beta_2(|\theta(t)|), \tag{50}$$

where for $i = 1, 2$, $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are continuous, strictly increasing functions with $\alpha_i(0) = \beta_i(0) = 0$, and $\lim_{r\to\infty} \alpha_i(r) = \lim_{r\to\infty} \beta_i(r) = \infty$.

(A3) From Eq. (38) (or Eq. (33)), the derivative of $\hat{V}$ satisfies

$$\dot{\hat{V}}(t) \leqslant -\alpha_3(\|z(t)\|_{\Im}) + \upsilon(t), \tag{51}$$

where $\alpha_3(\cdot)$ is a continuous, strictly increasing function with $\alpha_3(0) = 0$ and $\upsilon(t)$ is uniformly bounded by σ.

Using (A1)–(A3), assertions (i)–(iii) of the theorem are proved as follows:

(i) (A2) and (A3) establish that $z(t)$ and $\theta(t)$ are uniformly ultimately bounded. This proof is collected in Appendix A. Let the uniform bound of $z(t)$ and $\theta(t)$ be $\bar{d}$ and $\bar{d}'$, respectively.

(ii) As shown in Appendix A, the magnitude of $\bar{d}$ is dependent on that of σ, i.e., $\sigma \approx 0$ denotes $\bar{d} \approx 0$. Thus, if σ is sufficiently small, then it is guaranteed that $z(t)$ is uniformly ultimately bounded within an arbitrarily small neighborhood of zero. As mentioned in Remark 1, $\upsilon(t)$ can be pushed in an arbitrarily small boundedness region by making sufficiently small ε_d, δ_d and sufficiently large γ_d. This implies that σ can be made arbitrarily small near to zero by a suitable choice of ε_d, δ_d, and γ_d. As the result, the uniform ultimate boundedness region of $z(t)$ is made arbitrarily small near to zero.

(iii) Assertion (iii) can be proved by applying Theorem 1 in Ref. [25]. From Eq. (48), the solution $z(t)$ of Eq. (44) at time t starting with the initial state $z(s)$ at initial time s can be written as

$$z(t) = \Phi(t, s)z(s) + \int_s^t \Phi(t, \tau)F(\tau, z(\tau), \theta(\tau))\,d\tau. \tag{52}$$

Then, a two-parameter family of map $M(t,s)$ on $\Im$ can be defined from Eq. (52) as

$$M(t,s)z(s) \triangleq z(t,z(s),s), \quad 0 \leqslant s \leqslant t < \infty. \tag{53}$$

Then, by the uniqueness and continuous dependence of the solution on the triple $\{t, z(s), s\}$, the mapping $M(t,s)$ on $\Im$ becomes an evolution process [26, pp. 12,49]. Now, the integration of both sides of Eq. (51) from 0 to ∞ yields

$$\int_0^\infty \alpha_3(\|z(t)\|)\,\mathrm{d}t = \int_0^\infty \alpha_3(\|M(t,0)z_0\|)\,\mathrm{d}t \leqslant \hat{V}(0) - \hat{V}(\infty) + \int_0^\infty \upsilon(t)\,\mathrm{d}t < \infty, \tag{54}$$

where the initial time and the initial state are chosen to be zero, respectively. Note that $z(t) = M(t,0)z_0$ has been used in deriving the first equality in Eq. (54).

The remainder of the proof is the same as that of Theorem 1 in Ref. [25], and then it is obtained that $z(t) \to 0$ as $t \to \infty$ if $\int_0^\infty \upsilon(t)\,\mathrm{d}t < \infty$.

Remark 4. Consider the following positive definite functional which is equivalent to the mechanical energy of the uncontrolled span:

$$V_{uncontrolled}(t) = \frac{\alpha}{2}\int_l^{\bar{l}} \{\rho(v_s w_x + w_t)^2 + T_s w_x^2\}\mathrm{d}x + 2\beta\int_l^{\bar{l}} \rho x w_x(w_t + v_s w_x)\mathrm{d}x,$$

where $x = \bar{l}$ denotes the position of the support roll, i.e., the right boundary of the uncontrolled span part. From (13)–(14), (21)–(23), and Theorem 1, it can be easily concluded that the velocity and displacement of uncontrolled span part are uniform ultimate bounded as mentioned in Section 2.

5. Numerical simulations

The effectiveness of the proposed boundary control law is illustrated by numerical simulations using a finite difference method. The coefficient ρ and the external disturbance force $d(t)$ are treated as unknown but actual value of ρ is taken as $\rho = 2.7$ kg/m for simulation purpose. Other parameters of the axially moving string used for numerical simulations are: $c_v = 0.001$ N m^2 s, $v_s = 1$ m/s, $m_c = 10$ kg, $d_c = 0.25$ N/m/s, $T_s = 26{,}000 + 10\sin 2x + 10\cos t$ N, and $l = 15$ m of the total length 30 m of the string. Assume that the periodic excitation takes place at the right side boundary at the string, i.e., support roll, by $0.3\sin 2t$. The initial conditions are $w_0 = 0.5\sin 3\pi x$ and $w_{t0} = 0$.

The control parameters are selected as $\alpha = 1$, $\beta = 0.01$, $\gamma_0 = 0.1$, $\gamma_1 = 10$, $\delta_d = 0.01$, $\gamma_d = 100$, $\varepsilon_d = 0.1$, $k_1 = 4.4$, and $k_2 = 4$, from which the conditions in Theorem 1 are satisfied, i.e.,

$$\alpha > 2\beta l = 0.3, \quad \left(\alpha c_v + \beta\rho - \frac{2\beta c_v l}{\gamma_1}\right) \cong 0.022 > 0, \quad T_{s,min} > \rho,$$

$$\left\{\beta T_{s,min} - \left(\beta l + \frac{\alpha v_s}{2}\right)(T_s)_{x,max} - \frac{\alpha}{2}(T_s)_{t,max} - 2\beta c_v \gamma_1 l\right\} \cong 248 > 0,$$

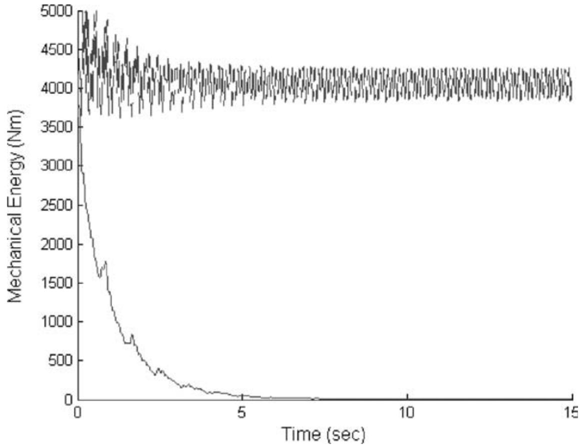

Fig. 2. Comparison of the controlled span part (solid line) and the uncontrolled span part (dashed line).

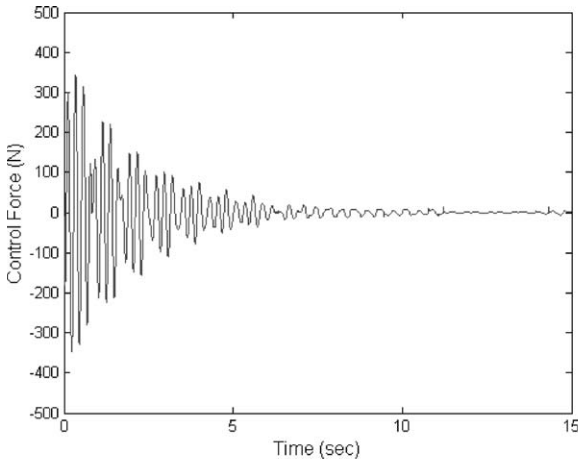

Fig. 3. The boundary input force used: $f_c(t)$.

$$0 < k_1 < \frac{T_s(l)\beta l - \beta\rho l v_s^2}{\alpha v_s + 2\beta l} = 2,997, \quad \text{and} \quad k_2 = \frac{k_1\alpha + 2\beta\rho l v_s}{\alpha v_s + 2\beta l} > \frac{\beta\rho l}{2\alpha} = 0.2,$$

where $T_{s,min} = 25,980$ N and $(T_s)_{x,max} = (T_s)_{t,max} = 10$ N.

A comparison of the mechanical energy (11) of the controlled span part and the uncontrolled span part of the string with the robust adaptive boundary force controller proposed is shown in Fig. 2. As analyzed in Section 4, the initial vibration energy of the controlled span part dissipates asymptotically with the control action, even though a local increase in the mechanical energy is shown due to the right boundary disturbance on the controlled span part as mentioned in Remark 3. However, the vibration energy of the uncontrolled span part remains almost at the same level. Fig. 3 shows the boundary control force. Figs. 4 and 5 depict the estimated values $\hat{\rho}$ and $\hat{\mu}_d$ of the system parameter ρ and the disturbance d, respectively, in which the estimated

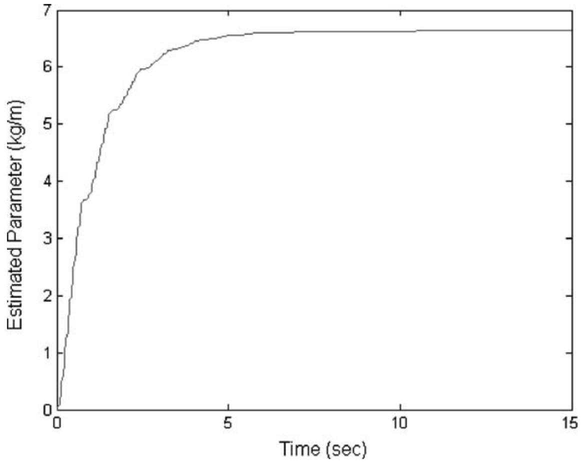

Fig. 4. Estimated parameter: $\hat{\rho}(t)$.

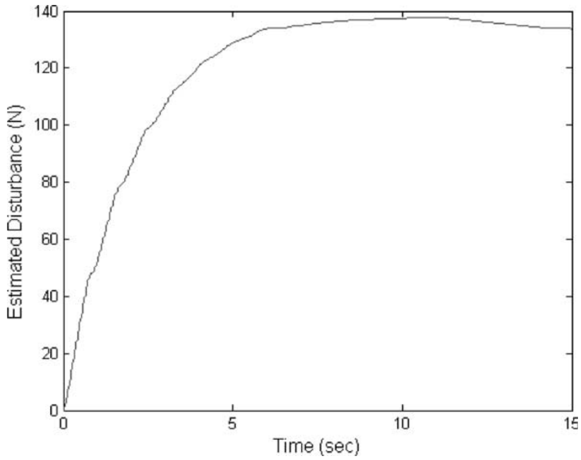

Fig. 5. Estimated parameter: $\hat{\mu}_d(t)$.

parameters converge to constant values, respectively, because $w_t(l)$ and $w_x(l)$ approach to zero. Note that it is not essential that the estimated parameters converge to the exact values in this control scheme.

6. Conclusions

In this paper, a robust adaptive boundary control scheme to suppress the transverse vibration of an axially moving string system under a spatiotemporally varying tension and an unknown boundary disturbance force has been investigated. The following findings are concluded.

(i) In the case of traveling continuous materials with a spatiotemporally varying tension, even if the displacements at both boundaries are zero, the time rate of the change of tension $T_s(x,t)$ can cause an increase of the total mechanical energy.

(ii) A robust adaptive boundary force controller can achieve the suppression of the vibration of the controlled span part of the string system in the presence of a spatiotemporally varying tension and an unknown boundary disturbance force while the vibrations of the uncontrolled span part remain. The asymptotic stability of the closed loop system with the robust boundary control law and the adaptation laws proposed have been proved with the Lyapunov method and semigroup theory.

(iii) Since the feedback terms in the robust boundary control law and the adaptation laws are the velocity, slope, and slope rate on the controlled side of the actuator, the vibration suppression of the controlled span part can be successfully implemented by the robust adaptive boundary controller.

Acknowledgements

This work was supported by the program of National Research Laboratory of the Ministry of Science and Technology, Korea.

Appendix A. Uniform ultimate boundedness [27]

Proof of Assertion (i) of Theorem 1. Consider the solutions of Eq. (44) given by $z(\cdot):[t_0,t_1]\to\Im$ and $\theta(\cdot):[t_0,t_1]\to R^2$ with initial conditions $\|z(t_0)\|\leqslant r$ and $|\theta(t_0)|\leqslant r'$, respectively. Let $\bar{r}\triangleq max\{r,R\}$, where $R\triangleq\alpha_3^{-1}(\sigma)$. Hence, $\|z(t_0)\|\leqslant\bar{r}$, $|\theta(t_0)|\leqslant\bar{r}'$, and $R\leqslant\bar{r}$. Let

$$d(\bar{r})=(\alpha_1^{-1}\circ\alpha_2)(\bar{r})\quad\text{and}\quad d(\bar{r}')=(\beta_1^{-1}\circ\beta_2)(\bar{r}'). \tag{A.1}$$

Furthermore, in view of Eq. (50),

$$\alpha_1(\bar{r})\leqslant\alpha_2(\bar{r})\quad\text{and}\quad\beta_1(\bar{r}')\leqslant\beta_2(\bar{r}'). \tag{A.2}$$

From Eqs. (A.1) and (A.2), $\bar{r}\leqslant d(\bar{r})$ and $\bar{r}'\leqslant d(\bar{r}')$.

Uniform boundedness

Now, show the uniform boundedness of $z(t)$ and $\theta(t)$ over $[t_0,t_1]$ by using a contradiction argument. Suppose there is a $t_3>t_0$ such that

$$\|z(t_3)\|>d(\bar{r})\quad\text{and}\quad|\theta(t_3)|>d(\bar{r}'). \tag{A.3}$$

Since $z(\cdot)$ and $\theta(\cdot)$ are continuous and

$$\|z(t_0)\|\leqslant\bar{r}\leqslant d(\bar{r})<\|z(t_3)\|\quad\text{and}\quad|\theta(t_0)|\leqslant\bar{r}'\leqslant d(\bar{r}')<|\theta(t_3)|$$

there exists a $t_2\in[t_0,t_3]$ such that $\|z(t_2)\|=\bar{r}$ and $|\theta(t_2)|=\bar{r}'$, and for $\forall t\in[t_2,t_3]$

$$\|z(t)\|\geqslant\bar{r}\geqslant R\quad\text{and}\quad|\theta(t)|\geqslant\bar{r}'. \tag{A.4}$$

Now using conditions (50), (51), and (A.4) yields

$$\alpha_1(\|z(t_3)\|) + \beta_1(|\theta(t_3)|) \leqslant V(t_3, z(t_3), \theta(t_3)) \leqslant \alpha_2(\bar{r}) + \beta_2(\bar{r}') + \int_{t_2}^{t_3} \{-\alpha_3(R) + \sigma\}\, \mathrm{d}\tau = \alpha_2(\bar{r}) + \beta_2(\bar{r}'), \tag{A.5}$$

where $R = \alpha_3^{-1}(\sigma)$ has been used in getting the last equality. Therefore, the fact that (A.5) should hold for both $z(\cdot)$ and $\theta(\cdot)$ implies

$$\alpha_1(\|z(t_3)\|) \leqslant \alpha_2(\bar{r}) \quad \text{and} \quad \beta_1(|\theta(t_3)|) \leqslant \beta_2(\bar{r}').$$

Hence, $\|z(t_3)\| \leqslant (\alpha_1^{-1} \circ \alpha_2)(\bar{r}) \leqslant d(\bar{r})$ and $|\theta(t_3)| \leqslant (\beta_1^{-1} \circ \beta_2)(\bar{r}') \leqslant d(\bar{r}')$. This now contradicts (A.3). Therefore, $\|z(t)\| \leqslant d(\bar{r})$ and $|\theta(t)| \leqslant d(\bar{r}')$.

Uniform ultimate boundedness

The uniform boundedness above is now extended over $[t_0, \infty)$. Let

$$\bar{d} > (\alpha_1^{-1} \circ \alpha_2)(R) \quad \text{and} \quad \bar{R} = (\alpha_2^{-1} \circ \alpha_1)(\bar{d})$$

so that $\bar{R} > R$ and $d(\bar{R}) = (\alpha_1^{-1} \circ \alpha_2)(\bar{R}) \triangleq \bar{d}$.

If $r \leqslant \bar{R}$ and $r' \leqslant \bar{R}'$, then $\|z(t_0)\| \leqslant \bar{R}$ and $|\theta(t_0)| \leqslant \bar{R}'$. Let $d(\bar{R}') = (\beta_1^{-1} \circ \beta_2)(\bar{R}') \triangleq \bar{d}'$. Hence, in view of the uniform boundedness results above, $\|z(t)\| \leqslant d(\bar{R}) = \bar{d}$ and $|\theta(t)| \leqslant d(\bar{R}') = \bar{d}'$ for $\forall t \in [t_0, \infty)$.

Next consider $r > \bar{R}$ and $r' > \bar{R}'$, and suppose that for $\forall t \in [t_0, t_1]$

$$\|z(t)\| > \bar{R} \quad \text{and} \quad |\theta(t)| > \bar{R}', \tag{A.6}$$

where $t_1 = t_0 + T(\bar{d}, \bar{d}', r, r')$ and

$$T(\bar{d}, \bar{d}', r, r') \triangleq \frac{\alpha_2(r) + \beta_2(r') - \alpha_1(\bar{R}) - \beta_1(\bar{R}')}{\alpha_3(\bar{R}) - \sigma}.$$

Then, using conditions (50), (51), and (A.6) yields

$$\alpha_1(\|z(t_1)\|) + \beta_1(|\theta(t_1)|) \leqslant V(t_1, z(t_1), \theta(t_1)) \leqslant \alpha_2(r) + \beta_2(r') + T(\bar{d}, \bar{d}', r, r')\{-\alpha_3(\bar{R}) + \sigma\} = \alpha_1(\bar{R}) + \beta_1(\bar{R}').$$

That is, $\|z(t_1)\| \leqslant \bar{R}$ and $|\theta(t_1)| \leqslant \bar{R}'$. But, this contradicts supposition (A.6). Hence, there must be a $t_2 \in [t_0, t_1]$ such that $\|z(t_2)\| \leqslant \bar{R}$ and $|\theta(t_2)| \leqslant \bar{R}'$. Then, as a consequence of the uniform boundedness result above, for $\forall t \geqslant t_2$

$$\|z(t)\| \leqslant d(\bar{R}) = \bar{d} \quad \text{and} \quad |\theta(t)| \leqslant d(\bar{R}') = \bar{d}'.$$

Hence, we have that $\|z(t)\| \leqslant \bar{d}$ and $|\theta(t)| \leqslant \bar{d}'$, $\forall t \geqslant t_0 + T(\bar{d}, \bar{d}', r, r')$.

Appendix B. Derivation of inequality (30)

Using Eqs. (27) and (28), the following is derived:

$$\begin{aligned}
&\bar{w}(l)(f_d + d) + \frac{1}{\gamma_d}\tilde{\mu}_d\dot{\hat{\mu}}_d \\
&\leqslant -\frac{\hat{\mu}_d^2}{\hat{\mu}_d|\bar{w}(l)| + \varepsilon_d}|\bar{w}(l)|^2 + \mu_d|\bar{w}(l)| - \frac{\delta_d}{\gamma_d}\hat{\mu}_d\tilde{\mu}_d + \tilde{\mu}_d|\bar{w}(l)| \\
&= \frac{-\hat{\mu}_d^2|\bar{w}(l)|^2 + \hat{\mu}_d^2|\bar{w}(l)|^2 + \hat{\mu}_d|\bar{w}(l)|\varepsilon_d}{\hat{\mu}_d|\bar{w}(l)| + \varepsilon_d} \\
&\quad - \frac{\delta_d}{2\gamma_d}\tilde{\mu}_d^2 - \left(\sqrt{\frac{\delta_d}{2\gamma_d}}\tilde{\mu}_d + \sqrt{\frac{\delta_d}{2\gamma_d}}\mu_d\right)^2 + \frac{\delta_d}{2\gamma_d}\mu_d \\
&\leqslant \varepsilon_d - \frac{\delta_d}{\gamma_d}\tilde{\mu}_d^2 + \frac{\delta_d}{2\gamma_d}\mu_d,
\end{aligned}$$

where $\bar{w}(l) \triangleq \{\alpha w_t(l) + (\alpha v_s + 2\beta\, l)w_x(l)\}$, and from which the inequality given in Eq. (30) is obtained.

Appendix C. Local Lipschitz continuity

Set $y = (w, \dot{w}, w(l), w_t(l), \hat{\rho}, \hat{\mu}_d)^{\mathrm{T}} \in W$ and $y' = (w', \dot{w}', w'(l), w_t'(l), \hat{\rho}', \hat{\mu}_d')^{\mathrm{T}} \in W$. Then,

$$\begin{aligned}
&\|F_0(t,y) - F_0(x,y')\|_W^2 \\
&= \frac{m_c\alpha}{2}|m_c^{-1}v_s(\hat{\rho} - \rho)(w_t(l) + v_s w_x(l)) - m_c^{-1}v_s(\hat{\rho}' - \rho)(w_t'(l) + v_s w_x'(l)) \\
&\quad + m_c^{-1}(f_d - d) - m_c^{-1}(f_d' - d)|^2 \\
&\quad + |\varepsilon\hat{\rho} - \varepsilon\hat{\rho}' + \gamma_0 v_s\{w_t(l) + v_s w_x(l)\}\{\alpha w_t(l) + (\alpha v_s + 2\beta l)w_x(l)\} \\
&\quad - \gamma_0 v_s\{w_t'(l) + v_s w_x'(l)\}\{\alpha w_t'(l) + (\alpha v_s + 2\beta l)w_x'(l)\}|^2 \\
&\quad + |\gamma_d|\alpha w_t(l) + (\alpha v_s + 2\beta l)w_x(l)| - \gamma_d|\alpha w_t'(l) + (\alpha v_s + 2\beta l)w_x'(l)||^2,
\end{aligned} \tag{C.1}$$

where

$$f_d'(t) = -\frac{\hat{\mu}_d'^2(t)}{\hat{\mu}_d'(t)|\bar{w}'(l)| + \varepsilon_d}\bar{w}'(l)$$

and $\bar{w}'(l) \triangleq \{\alpha w_t'(l) + (\alpha v_s + 2\beta l)w_x'(l)\}$.

Consider the following inequalities:

$$|w_x(l)w_t(l) - w_x'(l)w_t'(l)|^2 \leqslant w_t^2|w_x(l) - w_x'(l)|^2 + w_x'^2|w_t(l) - w_t'(l)|^2, \tag{C.2}$$

and

$$
\begin{aligned}
|f_d - f_d'|^2 &= \left| \frac{\hat{\mu}_d^2}{\hat{\mu}_d|\bar{w}(l)| + \varepsilon_d}\bar{w}(l) - \frac{\hat{\mu}_d'^2}{\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d}\bar{w}(l) + \frac{\hat{\mu}_d'^2}{\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d}\bar{w}(l) - \frac{\hat{\mu}_d'^2}{\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d}\bar{w}'(l) \right|^2 \\
&\leqslant \left| \frac{\hat{\mu}_d\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d(\hat{\mu}_d + \hat{\mu}')}{(\hat{\mu}_d|\bar{w}(l)| + \varepsilon_d)(\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d)} \right|^2 |\bar{w}(l)|^2 |\hat{\mu}_d - \hat{\mu}_d'|^2 \\
&\quad + \left| \frac{\varepsilon_d}{(\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d)(\hat{\mu}_d'|\bar{w}'(l)| + \varepsilon_d)} \right|^2 |\hat{\mu}_d'|^4 |\bar{w}(l) - \bar{w}'(l)|^2 \\
&\quad + 2\left| \frac{\hat{\mu}_d'}{(\hat{\mu}_d'|\bar{w}(l)| + \varepsilon_d)(\hat{\mu}_d'|\bar{w}'(l)| + \varepsilon_d)} \right|^2 |\hat{\mu}_d'|^4 |\bar{w}'(l)|^2 |\bar{w}(l) - \bar{w}'(l)|^2. \qquad \text{(C.3)}
\end{aligned}
$$

Therefore, the following inequality is easily derived from Eq. (C.1) by using Eqs. (16), (C.2), and (C.3):

$$
\|F_0(t, y) - F_0(t, y')\|_W \leqslant C_4 \|y - y'\|_W,
$$

where $y, y' \in W$, and C_4 is a positive constant.

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